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Sine-Gordon breather form factors and quantum field equations

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Abstract
Using the results of previous investigations on sine-Gordon form factors, exact expressions of all breather matrix elements are obtained for several operators: all powers of the fundamental Bose field, general exponentials of it, the energy–momentum tensor and all higher currents. Formulae for the asymptotic behaviour of bosonic form factors are presented which are motivated by Weinberg's power counting theorem in perturbation theory. It is found that the quantum sine-Gordon field equation holds, and an exact relation between the 'bare' mass and the renormalized mass is obtained. Also a quantum version of a classical relation for the trace of the energy–momentum is proved. The eigenvalue problem for all higher conserved charges is solved. All results are compared with perturbative Feynman graph expansions and full agreement is found.

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1. Introduction

This work continues previous investigations [1, 2] on exact form factors for the sine-Gordon, alias the massive Thirring, model. Some results of the present paper have been published previously [3]. The classical sine-Gordon model is given by the wave equation

$$\Box\varphi(t,x) + \frac{\alpha}{\beta}\sin\beta\varphi(t,x) = 0.$$

Since Coleman [4] found the wonderful duality between the quantum sine-Gordon and the massive Thirring model, a great deal of effort has been made to understand this quantum field theoretic model. The present article is a further contribution in this direction. The main new results are:

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(1) Using Weinberg's power counting theorem, we prove in perturbation theory that matrix elements of exponentials of a Bose field satisfy a 'cluster property' in momentum space. We use this as a characterizing property for exponentials of Bose fields.
(2) In [2] we introduced the concept of 'p-functions' which belong to local operators for the sine-Gordon solitons (see also [5, 6]). Here we formulate that concept for the sine-Gordon breathers.
(3) We investigate the higher conservation laws which are typical for integrable quantum field theories. Hereby, we correct a mistake in the literature (see footnote 8).
(4) We prove the quantum field equation of motion. Thus we derive independently the matrix elements of the operators φ and $:\sin\beta\varphi:$ and show that they satisfy the field equation (after a finite mass renormalization). Hereby, we again correct some mistakes in the literature (see footnote 9).

In addition, we also recall some known formulae in order to present a more complete picture of the sine-Gordon breather form factors. The sine-Gordon model, alias the massive Thirring model, describes the interaction of several types of particles: solitons, anti-solitons, alias fermions and anti-fermions, and a finite number of charge-less breathers, which may be considered as bound states of solitons and anti-solitons. Integrability of the model implies that the n-particle S-matrix factorizes into two-particle S-matrices.

The 'bootstrap' program for integrable quantum field theoretical models in 1 + 1 dimensions starts as the first step with the calculation of the S-matrix. Here (see e.g. [7, 8]) our starting point is the two-particle sine-Gordon S-matrix for the scattering of fundamental bosons (lowest breathers) [9][2],

$$S(\theta) = \frac{\sinh\theta + \mathrm{i}\sin\pi\nu}{\sinh\theta - \mathrm{i}\sin\pi\nu}.$$

The pole of $S(\theta)$ at $\theta = \mathrm{i}\pi\nu$ belongs to the second breather b_2 as a breather–breather bound state. The parameter ν is related to the sine-Gordon and the massive Thirring model coupling constant by

$$\nu = \frac{\beta^2}{8\pi - \beta^2} = \frac{\pi}{\pi + 2g}$$

where the second equation is due to Coleman [4].

As a second step of the 'bootstrap' program, off-shell quantities as arbitrary matrix elements of local operators

$${}^{\mathrm{out}}\langle p'_m, \ldots, p'_1|\mathcal{O}(x)|p_1, \ldots, p_n\rangle^{\mathrm{in}}$$

are obtained by means of the 'form factor program' from the S-matrix as an input. Form factors for an integrable model in 1 + 1 dimensions were first investigated by Vergeles and Gryanik [10] for the sinh-Gordon model and by Weisz [12][3] for the sine-Gordon model. The 'form factor program' was formulated in [14, 15] where the concept of generalized form factors was introduced. In that paper consistency equations were formulated which are expected to be satisfied by these objects. Thereafter this approach was developed further and studied in the context of several explicit models by Smirnov [16] who proposed the form factor equations (see below) as extensions of similar formulae in the original paper [14]. Further publications on form factors and in particular on sine-Gordon and sinh-Gordon form factors are [17–27, 30]. Smirnov's approach in [17] is similar to the one used in the present article (see section 6). Also, there is a nice application [28, 29] of form factors in condensed matter physics, for example, for one-dimensional Mott insulators.

[2] This S-matrix element has been discussed before in [10, 11].
[3] Similar results were obtained by Zamolodchikov [13].

Let $\mathcal{O}(x)$ be a local operator. The generalized form factors $\mathcal{O}_n(\theta_1, \ldots, \theta_n)$ are defined by the vacuum—n-lowest breather matrix elements

$$\langle 0|\mathcal{O}(x)|p_1, \ldots, p_n\rangle^{\text{in}} = \mathrm{e}^{-\mathrm{i}x(p_1+\cdots+p_n)}\mathcal{O}_n(\theta_1, \ldots, \theta_n) \qquad \text{for} \quad \theta_1 > \cdots > \theta_{\mathrm{n}}$$

where the θ_i are the rapidities of the particles $p_i^\mu = m(\cosh\theta_i, \sinh\theta_i)$. In the other sectors of the variables, the functions $\mathcal{O}_n(\theta_1, \ldots, \theta_n)$ are given by analytic continuation with respect to the θ_i. General matrix elements are obtained from $\mathcal{O}_n(\underline{\theta})$ by crossing, which means in particular the analytic continuation $\theta_i \to \theta_i \pm \mathrm{i}\pi$.

In [14] one of the present authors (MK) and Weisz showed that for the case of a diagonal S-matrix, the n-particle form factor may be written as

$$\mathcal{O}_n(\underline{\theta}) = K_n^{\mathcal{O}}(\underline{\theta}) \prod_{1\leqslant i<j\leqslant n} F(\theta_{ij}) \tag{1}$$

where $\theta_{ij} = |\theta_i - \theta_j|$ and $F(\theta)$ is the two-particle form factor (see section 2). The K-function is an even $2\pi\mathrm{i}$ periodic meromorphic function. In [2] we presented a general formula for soliton–anti-soliton form factors in terms of an integral representation. Using the bound state fusion method, we derived the general soliton breather and pure breather form factor formula which we will investigate in this article in more detail. In particular for the case of lowest breathers, the K-function turns out to be of the form[4]

$$K_n^{\mathcal{O}}(\underline{\theta}) = \sum_{l_1=0}^{1} \cdots \sum_{l_n=0}^{1} (-1)^{l_1+\cdots+l_n} \prod_{1\leqslant i<j\leqslant n} \left(1 + (l_i - l_j)\frac{\mathrm{i}\sin\pi\nu}{\sinh\theta_{ij}}\right) p_n^{\mathcal{O}}(\underline{\theta}, \underline{l}). \tag{2}$$

The breather p-function $p_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n; l_1, \ldots, l_n)$ encodes the dependence on the operator $\mathcal{O}(x)$. It is obtained from the solitonic p-function $p_{\text{sol},2n}^{\mathcal{O}}(\tilde{\theta}_1, \ldots, \tilde{\theta}_{2n}; z_1, \ldots, z_n)$ (see [2]) by setting $\tilde{\theta}_{2i-1} = \theta_i + \frac{1}{2}\mathrm{i}u^{(1)}, \tilde{\theta}_{2i} = \theta_i - \frac{1}{2}\mathrm{i}u^{(1)}$ and $z_j = \theta_j - \frac{\mathrm{i}\pi}{2}(1 - (-1)^{l_j}\nu)$ with the fusion angle $u^{(1)} = \pi(1-\nu)$. In [2] we proposed the solitonic p-functions for several local operators. In this way, we obtained all breather p-functions which are some sort of decedents of solitonic p-functions, i.e. we just used the solitonic ones in the bound state points. In the present article we will take a somewhat different point of view. We will obtain a wider class of p-functions corresponding to local operators with respect to the breather field, also including operators which are non-local with respect to the solitonic field. The alternative point of view is the following: as already mentioned, it has been shown in [14] that a form factor of n fundamental bosons (lowest breathers) is of the form (1) where the K-function $K_n^{\mathcal{O}}(\underline{\theta})$ is meromorphic, symmetric and periodic (under $\theta_i \to \theta_i + 2\pi\mathrm{i}$). In addition, it has to satisfy some additional conditions (see section 2). We consider equation (2) as an ansatz for the K-function which transforms these conditions on the K-functions into simple equations for the p-functions. In section 4 we present solutions of these equations.

In this paper, we propose the p-functions for several local operators. In particular, we consider the infinite set of local currents $J_L^{\pm}(x)$, $L = \pm 1, \pm 3, \ldots$ belonging to the infinite set of conservation laws which are typical for integrable quantum field theories. For this example, the correspondences between the operators, the K-functions and the p-functions are (up to normalization constants)

$$J_L^{\pm}(x) \leftrightarrow K_n^{(L,\pm)}(\underline{\theta}) \leftrightarrow p_n^{(L,\pm)}(\underline{\theta}, \underline{l}) \propto \sum_{i=1}^{n} \mathrm{e}^{\pm\theta_i} \sum_{i=1}^{n} \mathrm{e}^{L(\theta_i - \frac{\mathrm{i}\pi}{2}(1-(-1)^{l_i}\nu))}.$$

Here the breather p-function is obtained from the corresponding solitonic one [2].

[4] For the sinh-Gordon model and the case of the exponential field, Brazhnikov and Lukyanov [23] found by different methods a formula which agrees with our results. Smirnov [17] derived an integral representation of sine-Gordon breather form factors (see section 6.2) which agrees for some cases with our results (e.g. for the current and the energy–momentum tensor), but not in the case of the higher currents (see also footnote 7).

In contrast to this case, the breather p-function for exponentials $:\mathrm{e}^{\mathrm{i}\gamma\varphi}:(x)$ of the field φ for generic real γ is not related to a solitonic p-function of any local operator (which means that $:\mathrm{e}^{\mathrm{i}\gamma\varphi}:(x)$ is not local with respect to the soliton field). Now the correspondences are (see footnote 4)

$$:\mathrm{e}^{\mathrm{i}\gamma\varphi}:(x) \leftrightarrow K_n^{(q)}(\underline{\theta}) \leftrightarrow p_n^{(q)}(\underline{l}) = N_n^{(q)} \prod_{i=1}^{n} q^{(-1)^{l_i}}$$

where $q = q(\gamma)$ (see section 4). Here and in the following $:\cdots:$ denotes normal ordering with respect to the physical vacuum. This, in particular for the vacuum expectation value, means $\langle 0| : \varphi^N : (x)|0\rangle = 0$ and therefore $\langle 0| : \mathrm{e}^{\mathrm{i}\gamma\varphi} : (x)|0\rangle = 1$. In section 4 we present arguments to support these correspondences and also determine the normalization constants $N_n^{\mathcal{O}}$.

As an application of these results we investigate quantum operator equations. In particular, we provide exact expressions for all matrix elements of all powers of the fundamental Bose field $\varphi(t, x)$ and its exponential $:\mathrm{e}^{\mathrm{i}\gamma\varphi}:(t, x)$ for arbitrary γ. We find that the operator $\Box^{-1} :\sin\beta\varphi(x):$ is local. Moreover, the quantum sine-Gordon field equation[5]

$$\Box\varphi(x) + m^2 \frac{\pi\nu}{\sin\pi\nu}\frac{1}{\beta} :\sin\beta\varphi:(x) = 0$$

is fulfilled for all matrix elements. The factor $\frac{\pi\nu}{\sin\pi\nu}$ modifies the classical equation and has to be considered as a quantum correction of the breather mass m as compared with the 'bare' mass $\sqrt{\alpha}$. Further, we find that the trace of the energy–momentum tensor $T^{\mu\nu}$ satisfies

$$T_\mu^\mu(x) = -2\frac{\alpha}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right)(:\cos\beta\varphi:(x) - 1).$$

Again this operator equation is modified by a quantum correction $(1 - \beta^2/8\pi)$ compared to the classical one.

We also show that the higher local currents $J_M^\mu(t, x)$ satisfy $\partial_\mu J_M^\mu(t, x) = 0$ and calculate all matrix elements of all higher conserved $Q_L = \int \mathrm{d}x\, J_L^0(t, x)$,

$$Q_L|p_1, \ldots, p_n\rangle^{\mathrm{in}} = \sum_{i=1}^{n} \mathrm{e}^{L\theta_i}|p_1, \ldots, p_n\rangle^{\mathrm{in}}. \tag{3}$$

In particular for $L = \pm 1$ the currents yield the energy–momentum tensor $T^{\mu\nu} = T^{\nu\mu}$ with $\partial_\mu T^{\mu\nu} = 0$.

The paper is organized as follows. In section 2 we recall some formulae of [1, 2] and in particular those for breather form factors, which we need in the following. The properties of the form factors are translated to conditions for the 'K-functions' and finally to simple ones of the 'p-functions'. In section 3 we investigate the asymptotic behaviour of bosonic form factors. In section 4 we discuss several explicit examples of local operators as general exponentials of the fundamental bose field, powers of the field, all higher conserved currents and the energy–momentum tensor. Using induction and Liouville's theorem we prove some identities, which means that the same operators may be represented in terms of different p-functions. These results are used in section 5 to prove operator field equations as the quantum sine-Gordon field equation. In section 6 we present further types of representations of sine-Gordon breather form factors: a determinant formula (see also [17, 19–21]) and two integral representations. One of them is new and could presumably be generalized to other models with no backward scattering (see also [31]). A proof is given in the appendix.

[5] In the framework of constructive quantum field theory, quantum field equations were considered in [32, 33]. For the sine-Gordon model, quantum field equations were discussed by Smirnov in [17] and for the sinh-Gordon model in [22] (see also footnote 9).

2. Breather form factors

Using the bound state fusion method, we derived in [2] from a general formula for soliton–anti-soliton form factors the pure breather form factor formula which in particular for the case of lowest breathers may be written in the form (1) with $F(\theta)$ being the two-particle form factor function. It satisfies Watson's equations

$$F(\theta) = F(-\theta)S(\theta) = F(2\pi\mathrm{i} - \theta)$$

with the S-matrix given above. Explicitly, it is given by the integral representation [14]

$$F(\theta) = \exp\left\{\int_0^\infty \frac{\mathrm{d}t}{t} \frac{\cosh\left(\frac{1}{2}+\nu\right)t - \cosh\frac{1}{2}t}{\cosh\frac{1}{2}t\,\sinh t} \cosh t\left(1 - \frac{\theta}{\mathrm{i}\pi}\right)\right\} \tag{4}$$

normalized such that $F(\infty) = 1$. In general, form factors of one kind of bosonic particle (i.e. with a diagonal S-matrix) satisfy the following properties [1, 14, 16].

2.1. Properties of the form factors

The form factor function $\mathcal{O}_n(\underline{\theta})$ is meromorphic with respect to all variables $\theta_1, \ldots, \theta_n$. It satisfies Watson's equations

$$\mathcal{O}_n(\ldots, \theta_i, \theta_j, \ldots) = \mathcal{O}_n(\ldots, \theta_j, \theta_i, \ldots)\, S(\theta_{ij}). \tag{5}$$

The crossing relation for the connected part (see e.g. [2]) of the matrix element means

$$\langle p_1|\mathcal{O}(0)|p_2, \ldots, p_n\rangle^{\mathrm{in}}_{\mathrm{conn}} = \mathcal{O}_n(\theta_1 + \mathrm{i}\pi, \theta_2, \ldots, \theta_n) = \mathcal{O}_n(\theta_2, \ldots, \theta_n, \theta_1 - \mathrm{i}\pi)$$

which implies in particular

$$\mathcal{O}_n(\theta_1, \theta_2, \ldots, \theta_n) = \mathcal{O}_n(\theta_2, \ldots, \theta_n, \theta_1 - 2\pi\mathrm{i}). \tag{6}$$

The function $\mathcal{O}_n(\underline{\theta})$ has poles determined by one-particle states in each sub-channel. In particular, it has the so-called annihilation poles at, for example, $\theta_{12} = \mathrm{i}\pi$ such that the recursion formula[6] is satisfied,

$$\operatorname*{Res}_{\theta_{12}=i\pi} \mathcal{O}_n(\theta_1, \ldots, \theta_n) = 2\mathrm{i}\mathcal{O}_{n-2}(\theta_3, \ldots, \theta_n)(\mathbf{1} - S(\theta_{2n}), \ldots, S(\theta_{23})). \tag{7}$$

Since we are dealing with relativistic quantum field theories, Lorentz covariance in the form

$$\mathcal{O}_n(\theta_1, \ldots, \theta_n) = \mathrm{e}^{-s\mu}\mathcal{O}_n(\theta_1 + \mu, \ldots, \theta_n + \mu) \tag{8}$$

holds if the local operator transforms as $\mathcal{O} \to \mathrm{e}^{s\mu}\mathcal{O}$ where s is the 'spin' of $\mathcal{O}$.

2.2. Conditions on the K-functions

Form factors of one kind of bosonic particle (which means that there is no backward scattering) may be expressed by equation (1) in terms of the K-functions. Therefore, properties of the form factors can be transformed into the following relations,

$$K_n^{\mathcal{O}}(\ldots, \theta_i, \theta_j, \ldots) = K_n^{\mathcal{O}}(\ldots, \theta_j, \theta_i, \ldots) \tag{9}$$

$$K_n^{\mathcal{O}}(\underline{\theta}) = K_n^{\mathcal{O}}(\theta_1 - 2\pi\mathrm{i}, \theta_2, \ldots, \theta_n) \tag{10}$$

$$\operatorname*{Res}_{\theta_{12}=\mathrm{i}\pi} K_n^{\mathcal{O}}(\underline{\theta}) = \frac{2\mathrm{i}}{F(\mathrm{i}\pi)} \prod_{i=3}^n \frac{1}{F(\theta_{2i} + \mathrm{i}\pi)F(\theta_{2i})} \left(1 - \prod_{i=3}^n S(\theta_{2i})\right) K_{n-2}^{\mathcal{O}}(\underline{\theta}'') \tag{11}$$

$$K_n^{\mathcal{O}}(\underline{\theta}) = \mathrm{e}^{-s\mu} K_n^{\mathcal{O}}(\theta_1 + \mu, \ldots, \theta_n + \mu) \tag{12}$$

where $\underline{\theta} = \theta_1, \ldots, \theta_n$ and $\underline{\theta}'' = \theta_3, \ldots, \theta_n$.

[6] This formula has been proposed in [16] as a generalization of formulae in [14] and it has been proved in [1] using LSZ assumptions.

2.3. Equations for the p-functions

Starting with a general integral representation for solitonic form factors and using the bound state fusion method, we have shown in [2] that the lowest breather K-functions may be expressed by equation (2) in terms of breather p-functions which follow from solitonic p-functions. As already mentioned in the introduction, we make the ansatz that the K-function is of the form (2) and we allow more general breather p-functions. Ansatz (2) transforms the conditions on the K-function $K_n^{\mathcal{O}}(\underline{\theta})$ into simpler equations for the p-function $p_n^{\mathcal{O}}(\underline{\theta},\underline{l})$. The p-function $p_n^{\mathcal{O}}(\underline{\theta},\underline{l})$ is holomorphic with respect to all variables $\theta_1,\ldots,\theta_n$, is symmetric with respect to the exchange of the variables θ_i and l_i at the same time, and is periodic with period $2\pi\mathrm{i}$:

$$p_n^{\mathcal{O}}(\ldots,\theta_i,\theta_j,\ldots,l_i,l_j,\ldots) = p_n^{\mathcal{O}}(\ldots,\theta_j,\theta_i,\ldots,l_j,l_i,\ldots) \tag{13}$$

$$p_n^{\mathcal{O}}(\underline{\theta},\underline{l}) = p_n^{\mathcal{O}}(\theta_1 - 2\pi\mathrm{i},\theta_2,\ldots,\theta_n,\underline{l}). \tag{14}$$

With the shorthand notation $\underline{\theta}' = \theta_2,\ldots,\theta_n$, $\underline{\theta}'' = \theta_3,\ldots,\theta_n$ and $\underline{l}'' = l_3,\ldots,l_n$ the recursion relation

$$p_n^{\mathcal{O}}(\theta_2 + \mathrm{i}\pi,\underline{\theta}',\underline{l}) = g(l_1,l_2)\,p_{n-2}^{\mathcal{O}}(\underline{\theta}'',\underline{l}'') + h(l_1,l_2) \tag{15}$$

holds where $g(0,1) = g(1,0) = 2/(F(\mathrm{i}\pi)\sin\pi\nu)$ and $h(l_1,l_2)$ is independent of $\underline{l}''$. Lorentz covariance reads as

$$p_n^{\mathcal{O}}(\theta_1+\mu,\ldots,\theta_n+\mu,\underline{l}) = \mathrm{e}^{s\mu}\,p_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n,\underline{l}). \tag{16}$$

We now show that these conditions of the p-function are sufficient to guarantee the properties of the form factors.

Theorem 1. *If the p-function $p_n^{\mathcal{O}}(\underline{\theta},\underline{l})$ satisfies the conditions (13)–(16), the K-function $K_n^{\mathcal{O}}(\underline{\theta})$ satisfies the conditions (9)–(12), and therefore the form factor function $\mathcal{O}_n(\underline{\theta})$ satisfies the properties (5)–(8).*

Proof. Except for (11) all claims are obvious. Taking the residue of (2) and inserting (11) we obtain (with $a = \mathrm{i}\sin\pi\nu$)

$$\begin{aligned}
\operatorname*{Res}_{\theta_{12}=\mathrm{i}\pi} K_n(\underline{\theta}) &= -a\sum_{l_3=0}^{1}\cdots\sum_{l_r=0}^{1}(-1)^{l_3+\cdots+l_n}\prod_{3=i<j}^{n}\left(1+\frac{l_i-l_j}{\sinh\theta_{ij}}a\right)\\
&\quad\times\sum_{l_1\neq l_2}(-1)^{l_1+l_2}(l_1-l_2)\prod_{i=3}^{n}\left(\left(1+\frac{l_1-l_i}{\sinh(\theta_{2i}+\mathrm{i}\pi)}a\right)\left(1+\frac{l_2-l_i}{\sinh\theta_{2i}}a\right)\right)\\
&\quad\times\left(g(l_1,l_2)\,p_{n-2}^{\mathcal{O}}(\underline{\theta}'',\underline{l}'')+h(l_1,l_2)\right)\\
&= \frac{2\mathrm{i}}{F(\mathrm{i}\pi)}K_{n-2}(\underline{\theta}'')\left(\prod_{i=3}^{n}\left(1+\frac{a}{\sinh\theta_{2i}}\right)-\prod_{i=3}^{n}\left(1-\frac{a}{\sinh\theta_{2i}}\right)\right)+h\text{-term}.
\end{aligned}$$

We have used the identity

$$\begin{aligned}
&\sum_{l_1\neq l_2}(-1)^{l_1+l_2}(l_1-l_2)\prod_{i=3}^{n}\left(\left(1+\frac{l_1-l_i}{\sinh(\theta_{2i}+\mathrm{i}\pi)}a\right)\left(1+\frac{l_2-l_i}{\sinh\theta_{2i}}a\right)\right)g(l_1,l_2)\\
&\qquad= g(0,1)\prod_{i=3}^{n}\left(1+\frac{a}{\sinh\theta_{2i}}\right)-g(1,0)\prod_{i=3}^{n}\left(1-\frac{a}{\sinh\theta_{2i}}\right)
\end{aligned}$$

valid for all l_i ($i \geqslant 3$). The same relation is valid when we replace g by h. The equation (11) now follows from $g(0,1) = g(1,0) = 2/(F(\mathrm{i}\pi)\sin\pi\nu)$ and

$$F(\theta + \mathrm{i}\pi)F(\theta) = 1 \bigg/ \left(1 - \frac{\mathrm{i}\sin\pi\nu}{\sinh\theta}\right)$$

(which is easily obtained from the integral representation (4)), provided that the h-term vanishes. This is a consequence of the following lemma. Note that the h-term is proportional to a $K_{n-2}(\underline{\theta}'')$ given by the formula (2) with a p-function independent of the $\underline{l}''$. □

Lemma 2. *If the p-function in (2) does not depend on $l_1, \ldots, l_n$ then the corresponding K-function vanishes.*

Proof. The proof is easy and obtained by using induction and Liouville's theorem: we easily obtain $K_1(\underline{\theta}) = K_2(\underline{\theta}) = 0$. As induction assumptions we take $K_{n-2}(\underline{\theta}'') = 0$. The function $K_n(\underline{\theta})$ is a meromorphic function in terms of the $x_i = \mathrm{e}^{\theta_i}$ with at most simple poles at $x_i = \pm x_j$ since $\sinh\theta_{ij} = (x_i + x_j)(x_i - x_j)/(2x_i x_j)$. The residues of the poles at $x_i = x_j$ vanish because of the symmetry. Furthermore, the residues at $x_i = -x_j$ are proportional to $K_{n-2}(\underline{\theta}'')$ because similar to the proof of theorem 1 we have

$$\operatorname*{Res}_{\theta_{12}=\mathrm{i}\pi} K_n(\underline{\theta}) = aK_{n-2}(\underline{\theta}'')\left(\prod_{i=3}^{n}\left(1 + \frac{a}{\sinh\theta_{2i}}\right) - \prod_{i=3}^{n}\left(1 - \frac{a}{\sinh\theta_{2i}}\right)\right).$$

Therefore the function $K_n(\underline{\theta})$ is holomorphic everywhere. Furthermore, for $x_1 \to \infty$ we have the asymptotic behaviour

$$K_n(\underline{\theta}) = \sum_{l_2=0}^{1}\cdots\sum_{l_n=0}^{1}(-1)^{l_2+\cdots+l_n}\prod_{2\leqslant i<j\leqslant n}\left(1 + (l_i - l_j)\frac{\mathrm{i}\sin\pi\nu}{\sinh\theta_{ij}}\right)$$
$$\times\sum_{l_1=0}^{1}(-1)^{l_1}\prod_{j=2}^{n}\left(1 + (l_1 - l_j)\frac{\mathrm{i}\sin\pi\nu}{\sinh\theta_{1j}}\right) \to 0. \tag{17}$$

Therefore $K_n(\underline{\theta})$ vanishes identically by Liouville's theorem. □

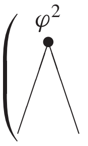
3. Asymptotic behaviour of bosonic form factors

In this section, we derive the asymptotic behaviour of bosonic form factors by means of general techniques of renormalized local quantum field theory. In particular, we use perturbation theory in terms of Feynman graphs. As the simplest example, we investigate first the asymptotic behaviour for $p_1 \to \infty$ or $\theta_{12} \to \infty$ of

$$\langle 0|:\varphi^2:|p_1, p_2\rangle^{\mathrm{in}} = 2\langle 0|\varphi(0)|p_1\rangle\langle 0|\varphi(0)|p_2\rangle + o(1)$$
$$= 2Z^{\varphi} + o(1)$$

where $:\cdots:$ means normal ordering with respect to the physical vacuum. This may be seen in perturbation theory as follows. Feynman graph expansion in lowest order means

$$\langle 0|:\varphi^2:|p_1, p_2\rangle^{\mathrm{in}} = 2\Bigg($$



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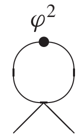
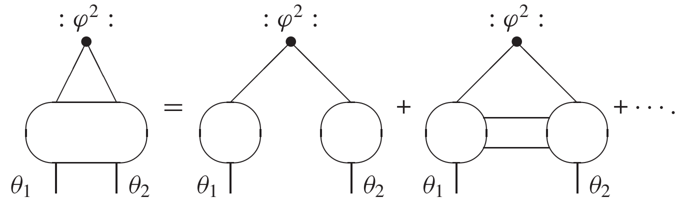


$$\Bigg) + O(\beta^4)$$

$$= 2\left(1 + \mathrm{i}\alpha\beta^2\frac{1}{2}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\mathrm{i}}{k^2 - m_1^2}\frac{\mathrm{i}}{(p-k)^2 - m_1^2}\right) + O(\beta^4)$$
$$= 2 + \frac{\beta^2}{4\pi}\frac{\mathrm{i}\pi - \theta_{12}}{\sinh\theta_{12}} + O(\beta^4).$$

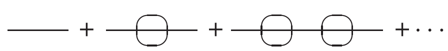
The second graph is of order $O(\ln p_1/p_1)$ for $p_1 \to \infty$. This is typical also for all orders in perturbation theory:



Weinberg's power counting theorem says that the second term and also all higher terms where more lines connect the two bubbles are again at least of order $O(\ln p_1/p_1)$ for $p_1 \to \infty$.

The wavefunction renormalization constant Z^φ is defined as usual by the two-point function of the (unrenormalized) field

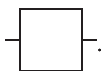
$$\int \langle 0|T\varphi(x)\varphi(0)|0\rangle \, \mathrm{e}^{\mathrm{i}px} \, \mathrm{d}^2x =$$



$$= \frac{\mathrm{i}}{p^2 - \alpha - \Pi(p^2)} = \frac{\mathrm{i}Z^\varphi}{p^2 - m^2 - \Pi_{\mathrm{ren}}(p^2)}$$

where $\Pi(p)$ is the self-energy, which means that it is given by the sum of all amputated one-particle irreducible graphs

$$-\mathrm{i}\Pi(p^2) =$$



The physical breather mass m, the wavefunction renormalization constant Z^φ and the renormalized breather self-energy are given by

$$m^2 = \alpha + \Pi(m^2)$$
$$\frac{1}{Z^\varphi} = 1 - \Pi'(m^2)$$
$$\Pi_{\mathrm{ren}}(p^2) = Z^\varphi(\Pi(p^2) - \Pi(m^2) - (p^2 - m^2)\Pi'(m^2)).$$

Since the sine-Gordon model is a 'super-renormalizable quantum field theory' both renormalization constants $\Pi(m^2)$ and $\Pi'(m^2)$ become finite after taking normal ordering in the interaction Lagrangian. They can be calculated exactly. The wavefunction renormalization constant was obtained in [14],

$$Z^\varphi = (1+\nu)\frac{\frac{\pi}{2}\nu}{\sin\frac{\pi}{2}\nu}\exp\left(-\frac{1}{\pi}\int_0^{\pi\nu}\frac{t}{\sin t}\,\mathrm{d}t\right) \tag{18}$$

and the relation of the unrenormalized and the physical mass is calculated in the present paper (see section 5)

$$\alpha = m^2\frac{\pi\nu}{\sin\pi\nu}.$$

Both relations have been checked in perturbation theory.

Remark 3. Usually in renormalized quantum field theory (in particular when Z is infinite) one would introduce the renormalized field with

$$\langle 0|\varphi_{\mathrm{ren}}(0)|p\rangle = 1$$

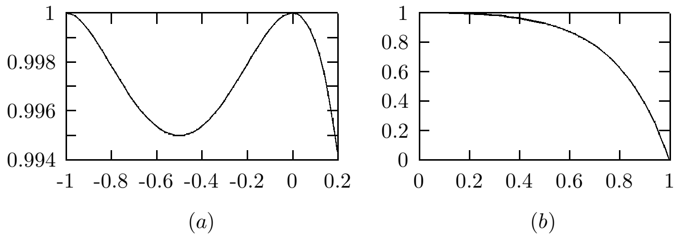


Figure 1. The wavefunction renormalization constant Z^φ given by equation (18) as a function of ν.

by $\varphi(x) = \sqrt{Z^\varphi}\varphi_{\rm ren}(x)$. Since Coleman's paper [4] however, the convention for the sine-Gordon model is to use the unrenormalized field $\varphi(x)$ which is related to the massive Thirring model current by

$$j^\mu = -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\varphi.$$

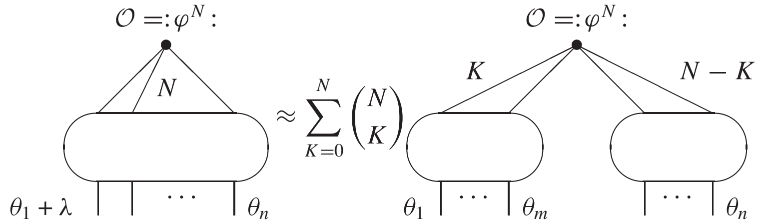
Therefore, we have the normalization

$$\langle 0|\varphi(0)|p\rangle = \sqrt{Z^\varphi}.$$

The wavefunction renormalization constant Z^φ is plotted as a function of ν in figure 1(a) for negative values of ν which correspond to the sinh-Gordon model and in (b) for $0 \leqslant \nu \leqslant 1$ which corresponds to the sine-Gordon model for $0 \leqslant \beta^2 \leqslant 4\pi$. Note that in ($a$) the function is symmetric with respect to the self-dual point $\nu = -\frac{1}{2}$ of the sinh-Gordon model and that in (b) $Z^\varphi = 1$ for the free breather point $\nu = 0$ and $Z^\varphi = 0$ for the free Fermi point $\nu = 1$ where the breather disappears from the particle spectrum.

As a generalization we now consider general n-particle form factors of a normal ordered arbitrary power of the field $\mathcal{O} =: \varphi^N:$ and let m of the momenta tend to infinity. If $\underline{\theta}(\lambda) = (\theta_1+\lambda, \ldots, \theta_m+\lambda, \theta_{m+1}, \ldots, \theta_n), \underline{\theta}' = (\theta_1, \ldots, \theta_m)$ and $\underline{\theta}'' = (\theta_{m+1}, \ldots, \theta_n)$ Weinberg's power counting theorem for bosonic Feynman graphs states that for $\mathrm{Re}\,\lambda \to \infty$

$$[\varphi^N]_n(\underline{\theta}(\lambda)) \approx \sum_{K=0}^{N}\binom{N}{K}[\varphi^K]_m(\underline{\theta}')[\varphi^{N-K}]_{n-m}(\underline{\theta}'') \tag{19}$$



with the notation $[\varphi^N]_n(\underline{\theta}) = \langle 0| : \varphi^N : (0)|p_1, \ldots, p_n\rangle^{\rm in}$. For the special case of a local operator which is an exponential of the fundamental Bose field $\mathcal{O} =: {\rm e}^{{\rm i}\gamma\varphi}:$ (for some γ) we therefore have

$$[{\rm e}^{{\rm i}\gamma\varphi}]_n\,(\underline{\theta}(\lambda)) = [{\rm e}^{{\rm i}\gamma\varphi}]_m\,(\underline{\theta}')\,[{\rm e}^{{\rm i}\gamma\varphi}]_{n-m}(\underline{\theta}'') + O({\rm e}^{-\lambda}).$$

This gives in particular for $m = 1$ and $\mathrm{Re}\,\theta_1 \to \infty$

$$[{\rm e}^{{\rm i}\gamma\varphi}]_n\,(\theta_1, \theta_2, \ldots, \theta_n) = [{\rm e}^{{\rm i}\gamma\varphi}]_1\,(\theta_1)\,[{\rm e}^{{\rm i}\gamma\varphi}]_{n-1}(\theta_2, \ldots, \theta_n) + O({\rm e}^{-\theta_1}). \tag{20}$$

4. Examples of operators

In this section, we present some examples of p-functions which satisfy the conditions of section 2, in particular (13)–(16), and propose the correspondence of local operators, K-functions and p-functions due to equations (1) and (2) for these examples,

$$\mathcal{O} \leftrightarrow K_n^{\mathcal{O}}(\underline{\theta}) \leftrightarrow p_n^{\mathcal{O}}(\underline{\theta}, \underline{l}).$$

4.1. Classical local operators

The classical sine-Gordon Lagrangian is

$$\mathcal{L}^{\mathrm{SG}} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{\alpha}{\beta^2}(\cos\beta\varphi - 1).$$

We consider the following classical local operators:

(1) $\mathrm{e}^{\mathrm{i}\gamma\varphi(x)}$ for arbitrary real γ.

(2) $\varphi^N(x)$.

(3) Higher conserved currents for $(L = 1, 3, 5\ldots)$

$$J_L^\rho = \begin{cases} J_L^+ = \partial^+\varphi(\partial^+)^L\varphi + O(\varphi^4) \\ J_L^- = ((\partial^+)^{L-1}\varphi + O(\varphi^2))\sin\varphi. \end{cases}$$

A second set of conserved currents is obtained by replacing $\partial^+ \to \partial^-$. The higher charges are of the form

$$Q_L = \int \mathrm{d}x(\partial^0\varphi\partial^{+L}\varphi + O(\varphi^4)) \qquad L = 1, 3, 5, \ldots$$

and the charges for even L vanish.

(4) $T^{\mu\nu}(x) = \partial^\mu\varphi\partial^\nu\varphi - g^{\mu\nu}\mathcal{L}^{\mathrm{SG}}$ the energy–momentum tensor or in terms of light cone coordinates ($\partial^\pm = \partial^0 \pm \partial^1$ etc)

$$T^{\pm\pm} = T^{00} \pm T^{01} \pm T^{10} + T^{11} = \begin{cases} \partial^+\varphi\partial^+\varphi \\ \partial^-\varphi\partial^-\varphi \end{cases}$$

$$T^{+-} = T^{00} - T^{01} + T^{10} - T^{11} = -2\frac{\alpha}{\beta^2}(\cos\beta\varphi - 1) = T^{-+}.$$

(5) $\mathrm{e}^{\mathrm{i}\beta\varphi(x)}$ for the particular value $\gamma = \beta$.

4.2. The normalization of form factors

The normalization constants are obtained in various cases by the following observations:

(a) The normalization of a field annihilating a one-particle state is given by the vacuum one-particle matrix element; in particular for the fundamental Bose field, one has

$$\langle 0|\varphi(0)|p\rangle = \sqrt{Z^\varphi}. \tag{21}$$

Z^φ is the finite wavefunction renormalization constant (18) which has been calculated in [14].

(b) If an observable like a charge $Q = \int \mathrm{d}x\mathcal{O}(x)$ belongs to a local operator, we use the relation

$$\langle p'|Q|p\rangle = q\langle p'|p\rangle.$$

This will be applied for example to the higher conserved charges.

(c) We use Weinberg's power counting theorem for bosonic Feynman graphs. As discussed in section 3, this yields in particular the asymptotic behaviour for the exponentials of the boson field $\mathcal{O} =: \mathrm{e}^{\mathrm{i}\gamma\varphi}:$,

$$\mathcal{O}_n(\theta_1, \theta_2, \ldots) = \mathcal{O}_1(\theta_1)\mathcal{O}_{n-1}(\theta_2, \ldots) + O(\mathrm{e}^{-\operatorname{Re}\theta_1})$$

as $\operatorname{Re}\theta_1 \to \infty$ in any order of perturbation theory. This behaviour is also assumed to hold for the exact form factors. Applying this formula iteratively, we obtain from (2) relations[7] for the normalization constants of the operators $: \mathrm{e}^{\mathrm{i}\gamma\varphi}:$

(d) The recursion relation (15) relates N_{n+2} and N_n. For all p-functions discussed below, this means

$$N_{n+2} = N_n \frac{2}{\sin\pi\nu F(\mathrm{i}\pi)} \qquad n \geqslant 1 \tag{22}$$

where $F(\mathrm{i}\pi)$ is related to the wavefunction renormalization constant by

$$\frac{1}{F(\mathrm{i}\pi)} = \frac{\beta^2}{8\pi\nu}\frac{\sin\pi\nu}{\pi\nu} Z^{\varphi}$$

see [2] and equation (18).

4.3. *Local operators and their p-functions*

For all cases to be discussed in the following, conditions (13)–(16) are again obvious except that of the recursion relation (15) which will be discussed in detail. For later convenience, we also list for some cases the explicit expressions of $K_n(\underline{\theta})$ for $n = 1, 2$ and the asymptotic behaviour of $K_n(\underline{\theta})$ for $\operatorname{Re}\theta_1 \to \infty$ which is easily obtained analogous to the calculation (17) in the proof of lemma 2. For convenience, we will use the notation $K_n = N_n \tilde{K}_n$ in the following.

4.3.1. Exponentials of the breather field. We propose the correspondence

$$\mathrm{e}^{\mathrm{i}\gamma\varphi} \leftrightarrow N_n^{(q)} \tilde{K}_n^{(q)}(\underline{\theta}) \leftrightarrow p_n^{(q)}(\underline{l}) = N_n^{(q)} \prod_{i=1}^{n} q^{(-1)^{l_i}} \tag{23}$$

with $q = q(\gamma)$ (and $q(0) = 1$) to be determined below. For low particle numbers, one easily calculates the K-functions

$$\tilde{K}_1^{(q)}(\theta) = (q - 1/q) \qquad \tilde{K}_2^{(q)}(\underline{\theta}) = (q - 1/q)^2 \tag{24}$$

and the asymptotic behaviour

$$\tilde{K}_n^{(q)}(\underline{\theta}) \approx \tilde{K}_1^{(q)}(\theta_1)\tilde{K}_{n-1}^{(q)}(\underline{\theta}').$$

The last formula is obtained similar to the proof of lemma 2. The proposal that the p-function $p_n^{(q)}(\underline{l})$ corresponds to an exponential of a bosonic field is supported by the following observation. The asymptotic behaviour is consistent with that of the form factors of exponentials of bosonic fields (20) as discussed in section 3. Indeed, it reads in terms of the K-functions as $K_n^{(q)}(\underline{\theta}) \approx K_1^{(q)}(\theta_1)K_{n-1}^{(q)}(\underline{\theta}')$ (since $F(\infty) = 1$) provided that the normalization constants satisfy

$$N_n^{(q)} = N_1^{(q)} N_{n-1}^{(q)} \quad \Rightarrow \quad N_n^{(q)} = \left(N_1^{(q)}\right)^n.$$

This is what we discussed above under point (c) to determine the normalization constants. Point (d) in the present case has the following meaning. The recursion condition (15) is

[7] This type of argument has also been used in [14, 20–22].

satisfied since in this case we have $g(l_1, l_2) = q^{(-1)^{l_1}+(-1)^{l_2}} N_n^{(q)} \big/ N_{n-2}^{(q)}$ which is symmetric and $h(l_1, l_2) = 0$. Condition (15) with $g(0, 1) = g(1, 0) = 2/(F(\mathrm{i}\pi)\sin\pi\nu)$ implies the recursion relation for the normalization constants (22) which finally yields

$$N_1^{(q)} = \sqrt{\frac{2}{F(\mathrm{i}\pi)\sin\pi\nu}} = \sqrt{Z^\varphi}\frac{\beta}{2\pi\nu} \qquad N_n^{(q)} = \left(\sqrt{Z^\varphi}\frac{\beta}{2\pi\nu}\right)^n \tag{25}$$

where Z^φ is the breather wavefunction renormalization constant (18). The relation of $F(\mathrm{i}\pi)$ with Z^φ is obtained by elementary manipulations of the integral representations (4) and (18). Recall that normal ordering implies $N_0^{(q)} = 1$.

4.3.2. Powers of the breather field. Motivated by the expansion of (23) with respect to $\ln q$ we propose the correspondence

$$\varphi^N \leftrightarrow N_n^{(N)} \tilde{K}_n^{(N)}(\underline{\theta}) \leftrightarrow p_n^{(N)}(\underline{l}) = N_n^{(N)} \left(\sum_{i=1}^n (-1)^{l_i}\right)^N. \tag{26}$$

Again one easily calculates $\left(\text{with } \tilde{K}_n^{(N)} = K_n^{(N)} \big/ N_n^{(N)}\right)$ the low particle number K-functions

$$\begin{aligned}
\tilde{K}_1^{(N)}(\theta) &= 2 \\
\tilde{K}_2^{(N)}(\underline{\theta}) &= 2^{N+1} \\
\tilde{K}_3^{(N)}(\underline{\theta}) &= 2(3^N - 3) - \sin^2\pi\nu \prod_{i<j} \frac{1}{\cosh\frac{1}{2}\theta_{ij}}
\end{aligned}$$

and the asymptotic behaviour

$$\tilde{K}_n^{(N)}(\underline{\theta}) \approx \sum_{K=1}^N \binom{N}{K} \tilde{K}_1^{(K)}(\theta_1) \tilde{K}_{n-1}^{(N-K)}(\underline{\theta}')$$

where $K_n^{(N)}$ is only nonvanishing for $N - n =$ even. This asymptotic behaviour agrees with (19) which follows from Weinberg's power counting argument and therefore justifies the correspondence (26). The normalization condition $\langle 0|\varphi(0)|p\rangle = \sqrt{Z^\varphi}$ (see (21)) yields

$$N_1^{(1)} = \tfrac{1}{2}\sqrt{Z^\varphi}.$$

The other normalization constants and also the function $q(\gamma)$ are now obtained as follows. Comparing the correspondences (23) and (26), we conclude

$$N_n^{(N)} = N_n^{(q)} \left(\frac{\ln q}{\mathrm{i}\gamma}\right)^N.$$

This implies for $N = n = 1$ together with (25)

$$q = \exp\left(\mathrm{i}\gamma \frac{N_1^{(1)}}{N_1^{(q)}}\right) = \exp\left(\mathrm{i}\frac{\pi\nu}{\beta}\gamma\right)$$

and finally the normalization constants

$$N_n^{(N)} = \left(\frac{1}{2}\sqrt{Z^\varphi}\right)^n \left(\frac{\pi\nu}{\beta}\right)^{N-n}. \tag{27}$$

We compare these general results with known special cases [14]. In particular for $n = N = 2$ we have

$$\begin{aligned}
\langle 0|{:}\varphi^2{:}(0)|p_1, p_2\rangle^{\mathrm{in}} &= K_2^{(2)}(\theta_{12})F(\theta_{12}) = 8N_2^{(2)}F(\theta_{12}) \\
&= 2Z^\varphi F(\theta_{12})
\end{aligned}$$

which agrees with formulae (4.4)–(4.6) of [14]. Further for $n = 3$ and $N = 1$ we have

$$\begin{aligned}\langle 0|\varphi(0)|p_1, p_2, p_3\rangle^{\text{in}} &= K_3^{(1)}(\underline{\theta})\prod_{i<j} F(\theta_{ij})\\ &= -(Z^\varphi)^{3/2}\frac{1}{8}\left(\beta\frac{\sin\pi\nu}{\pi\nu}\right)^2\prod_{i<j}\frac{F(\theta_{ij})}{\cosh\frac{1}{2}\theta_{ij}}\end{aligned}$$

which again agrees with formulae (4.9)–(4.12) of [14].

4.3.3. Higher conserved currents. In the following, we present new results concerning the higher conservation laws which are typical for integrable quantum field theories[8]. We propose the correspondence

$$J_L^\pm \leftrightarrow p_n^{(L,\pm)}(\underline{\theta}, \underline{l}) = \pm N_n^{(J_L)}\sum_{i=1}^n \mathrm{e}^{\pm\theta_i}\sum_{i=1}^n \mathrm{e}^{L(\theta_i - \frac{\mathrm{i}\pi}{2}(1-(-1)^{l_i}\nu))}$$

for n = even and zero for n = odd ($L = \pm 1, \pm 3, \ldots$). Again one easily calculates the two-particle K-function,

$$\tilde{K}_2^{(L,\pm)}(\underline{\theta}) = -2(-\mathrm{i})^L \sin\frac{1}{2}L\pi\nu\frac{\sin\pi\nu}{\sinh\theta_{12}}(\mathrm{e}^{\pm\theta_1} + \mathrm{e}^{\pm\theta_2})(\mathrm{e}^{L\theta_1} - \mathrm{e}^{L\theta_2}). \tag{28}$$

The recursion condition (15) is satisfied since $g(l_1, l_2) = N_n^{(J_L)}\big/N_{n-2}^{(J_L)}$ is symmetric and $h(l_1, l_2) = \sum_{i=3}^n \mathrm{e}^{\pm\theta_i}\sum_{i=1}^2 \mathrm{e}^{L(\theta_i - \frac{\mathrm{i}\pi}{2}(1-(-1)^{l_i}\nu))}$ is independent of l_i, $i > 2$. Again we have the recursion relation for the normalization constants (22). The two-particle normalization we calculated by means of (b) with the charges

$$\begin{aligned}\langle p'|Q_L|p\rangle &= \int_{-\infty}^{\infty}\mathrm{d}x\,\langle p'|\frac{1}{2}(J_L^+(x) + J_L^-(x))|p\rangle\\ &= \int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}(p-p')x}\frac{1}{2}\left(K_2^{(L,+)} + K_2^{(L,-)}\right)(\theta' + \mathrm{i}\pi, \theta)F(\theta' + \mathrm{i}\pi - \theta)\\ &= 2\pi\delta(p - p')\frac{1}{2}\left(K_2^{(L,+)} + K_2^{(L,-)}\right)(\theta + \mathrm{i}\pi, \theta)F(\mathrm{i}\pi)\\ &= \langle p'|p\rangle\,\mathrm{e}^{L\theta} \qquad \text{if } L \text{ odd.}\end{aligned}$$

Using (28) we obtain

$$\begin{aligned}\tfrac{1}{2}\left(K_2^{(L,+)} + K_2^{(L,-)}\right)(\theta + \mathrm{i}\pi, \theta) &= -N_n^{(J_L)}2\,(-\mathrm{i})^L \sin\tfrac{1}{2}L\pi\nu\sin\pi\nu\cosh\theta\,\mathrm{e}^{L\theta}(\mathrm{e}^{L\mathrm{i}\pi} - 1)\\ &= 2m\cosh\theta\,\mathrm{e}^{L\theta}/F(\mathrm{i}\pi)\end{aligned}$$

for L odd. For even L the charges vanish as in the classical case. With the relation of the normalization constants (22) we finally obtain

$$N_n^{(J_L)} = \frac{m\mathrm{i}^L}{4\sin\frac{1}{2}L\pi\nu}\left(\sqrt{Z^\varphi}\frac{\beta}{2\pi\nu}\right)^n. \tag{29}$$

Next we derive all eigenvalues of the higher charges (3). We show that for $n' + n > 2$ the connected part of the matrix element ${}^{\text{out}}\langle p_1', \ldots, p_{n'}'|Q_L|p_1, \ldots, p_n\rangle^{\text{in}}$ vanishes. The analytic

[8] In [17] form factors of higher currents in the sine-Gordon model were proposed, however the charges of these currents vanish. The densities were proposed to be of the form $(\partial^+)^L\partial^1 A(x)$ where the operator A is related to the energy–momentum tensor, in particular $\partial^0\partial^1 A = T^{10}$ and $\partial^1\partial^1 A = T^{00}$. For $L > 1$ obviously $\int \mathrm{d}x(\partial^+)^L\partial^1 A(x) = \int \mathrm{d}x(\partial^+)^{L-2}\partial^1(T^{10} + T^{00} + T^{01} + T^{00}) = 0$ where the conversation laws $\partial^0 T^{\mu 0} = \partial^1 T^{\mu 1}$ have been used.

continuation $\mathcal{O}_{n'+n}(\underline{\theta}'+\mathrm{i}\pi, \underline{\theta})$ yields this connected part. From the correspondence of operators and p-functions

$$Q_L = \int \mathrm{d}x\, J_L^0(x) \leftrightarrow 2\pi\delta(P'-P) N_n^{(J_L)} m \left(-\sum_{i=1}^{n'} \sinh\theta_i' + \sum_{i=1}^{n} \sinh\theta_i \right)$$
$$\times \left(\sum_{i=1}^{n'} \mathrm{e}^{L(\theta_i'+\mathrm{i}\pi - \frac{\mathrm{i}\pi}{2}(1-(-1)^{l_i}\nu))} + \sum_{i=1}^{n} \mathrm{e}^{L(\theta_i - \frac{\mathrm{i}\pi}{2}(1-(-1)^{l_i}\nu))} \right)$$

the claim follows since for $n'+n > 2$ there are no poles which may cancel the zero at $P' = P$ where $P^{(\prime)} = \sum p_i^{(\prime)}$. Therefore, contributions to the matrix element come from disconnected parts which contain (analytic continued) two-particle form factors,

$$
\begin{aligned}
&{}^{\mathrm{out}}\langle p_1', \ldots, p_{n'}' | Q_L | p_1, \ldots, p_n \rangle^{\mathrm{in}} \\
&\quad = \sum_{i,j} {}^{\mathrm{out}}\langle p_1', \ldots, \hat{p}_i', \ldots, p_{n'}' | p_1, \ldots, \hat{p}_j, \ldots, p_n \rangle^{\mathrm{in}} \, \langle p_i' | Q_L | p_j \rangle \\
&\quad = {}^{\mathrm{out}}\langle p_1', \ldots, p_{n'}' | p_1, \ldots, p_n \rangle^{\mathrm{in}} \sum_{i=1}^{n} \mathrm{e}^{\theta_i L}
\end{aligned}
$$

where $\hat{p}_j$ means that this particle is missing in the state.

From the higher currents for $L = \pm 1$ we get the light cone components of the energy–momentum tensor $T^{\rho\sigma} \propto J_\sigma^\rho$ with $\rho, \sigma = \pm$ (see also [22]).

4.3.4. The energy–momentum tensor. We propose the correspondence

$$T^{\rho\sigma} \leftrightarrow p_n^{\rho\sigma}(\underline{\theta}, \underline{l}) = \rho N_n^{(T)} \sum_{i=1}^{n} \mathrm{e}^{\rho\theta_i} \sum_{i=1}^{n} \mathrm{e}^{\sigma(\theta_i - \frac{\mathrm{i}\pi}{2}(1-(-1)^{l_i}\nu))} \tag{30}$$

for n even and $p_n^{\rho\sigma} = 0$ for n odd. The normalization is again determined by (c) namely

$$\langle p' | P^\nu | p \rangle = \langle p' | \int \mathrm{d}x\, T^{0\nu}(x) | p \rangle = \langle p' | p \rangle p^\nu \tag{31}$$

which in analogy to (29) gives

$$N_n^{(T)} = \frac{\mathrm{i}m^2}{4 \sin \frac{1}{2}\pi\nu} \left(\sqrt{Z^\varphi} \frac{\beta}{2\pi\nu} \right)^n . \tag{32}$$

Note that at first sight the energy–momentum tensor does not seem to be symmetric. However, it is due to an identity proved in the next section (see theorem 5). The conservation law follows as above for the higher currents and also the eigenvalue equation of the energy–momentum operator with the result

$$P^\nu | p_1, \ldots, p_n \rangle^{\mathrm{in}} = \sum_{i=1}^{n} p_i^\nu | p_1, \ldots, p_n \rangle^{\mathrm{in}} .$$

4.3.5. Special exponentials of the breather field. For the special cases of the exponential of the field $\gamma = \pm\beta$ we propose the alternative correspondence to (23),

$$\mathrm{e}^{\pm\mathrm{i}\beta\varphi} \leftrightarrow N_n^{\pm} \tilde{K}_n^{\pm}(\underline{\theta}) \leftrightarrow p_n^{\pm}(\underline{\theta}, \underline{l}) = N_n^{\pm} \sum_{i=1}^{n} \mathrm{e}^{\mp\theta_i} \sum_{i=1}^{n} \mathrm{e}^{\pm(\theta_i - \frac{\mathrm{i}\pi}{2}(1-(-1)^{l_i}\nu))} . \tag{33}$$

Again one easily calculates for low particle number the K-functions

$$\tilde{K}_1^{\pm}(\theta) = 2 \sin \tfrac{1}{2}\pi\nu \qquad \tilde{K}_2^{\pm}(\underline{\theta}) = \pm 4\mathrm{i} \sin \tfrac{1}{2}\pi\nu \sin \pi\nu$$

and the asymptotic behaviour

$$\tilde{K}_n^{\pm}(\underline{\theta}) \approx \pm 2\mathrm{i}\sin\pi\nu\,\tilde{K}_{n-1}^{\pm}(\underline{\theta}').$$

The proof of the last formula is given in the appendix. The normalization constants are calculated analogous to the case of the general exponential and take the form

$$N_n^{\pm} = \pm\mathrm{i}\frac{\sin\pi\nu}{\sin\frac{1}{2}\pi\nu}\left(\sqrt{Z^{\varphi}}\frac{\beta}{2\pi\nu}\right)^n. \tag{34}$$

4.4. Identities

It turns out that the correspondence between local operators and p-functions is not unique. In this section, we prove some identities which we will need in the following section to prove operator equations. To have a consistent interpretation of $K_n^{(q)}(\underline{\theta})$ with $q = \mathrm{e}^{\mathrm{i}\pi\nu\gamma/\beta}$ as the K-function of $\mathrm{e}^{\mathrm{i}\gamma\varphi(x)}$ it is necessary that $K_n^{(q)}(\underline{\theta})$ is even/odd for n = even/odd under the exchange $q \leftrightarrow 1/q$. For $\gamma = \pm\beta$ the K-functions of the general exponentials should turn into the K-functions of the special exponentials. These facts are expressed by the following lemma.

Lemma 4. *Let the K-functions*

$$K_n(\underline{\theta}) = \sum_{l_1=0}^{1}\cdots\sum_{l_r=0}^{1}(-1)^{l_1+\cdots+l_r}\prod_{1\leqslant i<j\leqslant n}\left(1+(l_i-l_j)\frac{\mathrm{i}\sin\pi\nu}{\sinh\theta_{ij}}\right)p_n(\underline{\theta},\underline{l})$$

be given by the p-functions,

$$K_n^{(q)}(\underline{\theta}) \leftrightarrow p_n^{(q)}(\underline{l}) = N_n^{(q)}\prod_{i=1}^{n}q^{(-1)^{l_i}}$$

$$K_n^{\pm}(\underline{\theta}) \leftrightarrow p_n^{\pm}(\underline{\theta},\underline{l}) = N_n^{\pm}\sum_{i=1}^{n}\mathrm{e}^{\mp\theta_i}\sum_{i=1}^{n}\mathrm{e}^{\pm(\theta_i-\frac{\mathrm{i}\pi}{2}(1-(-1)^{l_i}\nu))}$$

$$K_n^{(1)}(\underline{\theta}) \leftrightarrow p_n^{(1)}(\underline{l}) = N_n^{(1)}\sum_{i=1}^{n}(-1)^{l_i}.$$

Then the following identities hold (again with $K_n(\underline{\theta}) = N_n\tilde{K}_n(\underline{\theta})$),

$$\tilde{K}_n^{(q)}(\underline{\theta}) = -(-1)^n\tilde{K}_n^{(1/q)}(\underline{\theta}) \qquad \tilde{K}_n^{+}(\underline{\theta}) = -(-1)^n\tilde{K}_n^{-}(\underline{\theta}) \tag{35}$$

in particular for $\gamma = \beta$, i.e. $q = \mathrm{e}^{\mathrm{i}\pi\nu}$,

$$K_n^{+}(\underline{\theta}) = K_n^{(q)}(\underline{\theta})$$

and furthermore

$$\tilde{K}_n^{(1)}(\underline{\theta}) = \frac{1}{2\sin\frac{1}{2}\pi\nu}\left(\sum_{i=1}^{n}\mathrm{e}^{\theta_i}\sum_{i=1}^{n}\mathrm{e}^{-\theta_i}\right)^{-1}\left(\tilde{K}_n^{+}(\underline{\theta})+\tilde{K}_n^{-}(\underline{\theta})\right). \tag{36}$$

Proof. Again as in the proof of lemma 2 we use induction and Liouville's theorem. We introduce the differences

$$f_n(\underline{\theta}) = \tilde{K}_n^{(q)}(\underline{\theta}) + (-1)^n\tilde{K}_n^{(1/q)}(\underline{\theta})$$

or

$$f_n(\underline{\theta}) = \tilde{K}_n^{+}(\underline{\theta}) + (-1)^n\tilde{K}_n^{-}(\underline{\theta})$$

or

$$f_n(\underline{\theta}) = K_n^+(\underline{\theta}) - K_n^{(q=\exp(i\pi\nu))}(\underline{\theta})$$

or

$$f_n(\underline{\theta}) = \tilde{K}_n^{(1)}(\underline{\theta}) - \frac{1}{2\sin\frac{1}{2}\pi\nu}\left(\sum_{i=1}^n e^{\theta_i}\sum_{i=1}^n e^{-\theta_i}\right)^{-1}\left(\tilde{K}_n^+(\underline{\theta}) + \tilde{K}_n^-(\underline{\theta})\right).$$

Then the results of the previous subsection 4.3 imply in all cases $f_1(\theta) = f_2(\underline{\theta}) = 0$. As induction assumptions we take $f_{n-2}(\underline{\theta}'') = 0$. The functions $f_n(\underline{\theta})$ are meromorphic functions in terms of the $x_i = e^{\theta_i}$ with at most simple poles at $x_i = \pm x_j$ since $\sinh\theta_{ij} = (x_i + x_j)(x_i - x_j)/(2x_i x_j)$. The residues of the poles at $x_i = x_j$ vanish because of the symmetry and again the residues at $x_i = -x_j$ are proportional to $f_{n-2}(\underline{\theta}'')$ due to the recursion relation (11). Furthermore for $x_i \to \infty$ again $f_n(\underline{\theta}) \to 0$. Therefore $f_n(\underline{\theta})$ vanishes identically by Liouville's theorem for all n. For the last case of $f_n(\underline{\theta})$ it has been used that because of (35) for n even both $\tilde{K}^{\pm}$-terms cancel and that they are equal for odd n. Due to (33) $\tilde{K}_n^+$ is proportional to $\sum_{i=1}^n e^{-\theta_i}$ and $\tilde{K}_n^-$ is proportional to $\sum_{i=1}^n e^{\theta_i}$. Hence both are proportional to $\sum_{i=1}^n e^{-\theta_i}\sum_{i=1}^n e^{+\theta_i}$ which means that there are no extra poles at $\sum_{i=1}^n e^{\theta_i} = 0$ or $\sum_{i=1}^n e^{-\theta_i} = 0$. □

5. Operator equations

The classical sine-Gordon model is given by the wave equation

$$\Box\varphi(t,x) + \frac{\alpha}{\beta}\sin\beta\varphi(t,x) = 0.$$

If this is fulfilled we also have the relation for the trace of the energy–momentum tensor,

$$T_\mu^\mu(t,x) = -2\frac{\alpha}{\beta^2}(\cos\beta\varphi(t,x) - 1).$$

In this section, we construct the quantum version of these two classical equations. In the following $: \cdots :$ denotes normal ordering with respect to the physical vacuum which, in particular for the vacuum expectation value, means $\langle 0| : e^{i\gamma\varphi} : (t,x)|0\rangle = 1$. As consequences of the identities of section 4.4 we can prove quantum field equations.

Theorem 5. *The following operator equations are to be understood in terms of all their matrix elements.*

(1) For the exceptional value $\gamma = \beta$ the operator $\Box^{-1} : \sin\gamma\varphi : (t,x)$ is local and the quantum sine-Gordon field equation holds[9]*,*

$$\Box\varphi(t,x) + \frac{\alpha}{\beta} : \sin\beta\varphi : (t,x) = 0 \tag{37}$$

if the 'bare' mass $\sqrt{\alpha}$ is related to the renormalized one by[10]

$$\alpha = m^2\frac{\pi\nu}{\sin\pi\nu}. \tag{38}$$

Here m is the physical mass of the fundamental boson.

(2) The energy–momentum tensor is symmetric and its trace satisfies

$$T_\mu^\mu(t,x) = -2\frac{\alpha}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right)(:\cos\beta\varphi : (t,x) - 1). \tag{39}$$

[9] This field equation was also discussed in [17] and for the sinh-Gordon case in [22]. However, in these papers the relations of the bare and the renormalized masses differ from (38) and are not consistent with perturbation theory and the results of [34, 35].

[10] Before this formula was found in [34, 35] by different methods.

(3) For all higher currents the conservation laws hold,

$$\partial_\mu J_L^\mu(x) = 0 \qquad L = \pm 1, \pm 3, \dots.$$

Proof.

(1) From (33) we have the correspondence between operators and K-functions:

$$\Box^{-1} \sin\beta\varphi \leftrightarrow \frac{K_n^+(\underline{\theta}) - K_n^-(\underline{\theta})}{2\mathrm{i}\sum_{i=1}^n \mathrm{e}^{\theta_i}\sum_{i=1}^n \mathrm{e}^{-\theta_i}}.$$

As shown in the proof of lemma 4 there are no poles at $\sum_{i=1}^n \mathrm{e}^{\theta_i} = 0$ or $\sum_{i=1}^n \mathrm{e}^{-\theta_i} = 0$. Therefore $\Box^{-1}$: $\sin\beta\varphi$: is a local operator. Furthermore by equation (36)

$$\sum_{i=1}^n \mathrm{e}^{\theta_i}\sum_{i=1}^n \mathrm{e}^{-\theta_i} K_n^{(1)}(\underline{\theta}) = \frac{\pi\nu}{\beta\sin\pi\nu}\frac{1}{2\mathrm{i}}\left(K_n^+(\underline{\theta}) - K_n^-(\underline{\theta})\right)$$

where the normalizations (27) and (34) have been used. This means in particular that

$$\frac{N_n^{(1)}}{N_n^+}\frac{\mathrm{i}}{\sin\frac{1}{2}\pi\nu} = \frac{\pi\nu}{\beta\sin\pi\nu}.$$

In terms of operators, this is just the quantum sine-Gordon field equation. Comparing this result with the classical equation, we obtain the relation (38) between the bare mass and the physical mass.

(2) Using (30) and (33) we have the correspondence between operators and K-functions for n even:

$$T^{+-} \leftrightarrow N_n^{(T)}\tilde{K}_n^- \qquad T^{-+} \leftrightarrow -N_n^{(T)}\tilde{K}_n^+$$

$$T_\mu^\mu \leftrightarrow K_n^{(T)}(\underline{\theta}) = -\tfrac{1}{2}N_n^{(T)}\left(\tilde{K}_n^+ - \tilde{K}_n^-\right) \qquad \cos\beta\varphi - 1 \leftrightarrow \tfrac{1}{2}\left(K_n^+(\underline{\theta}) + K_n^-(\underline{\theta})\right).$$

The symmetry $T^{+-} = T^{-+}$ is again a consequence of (35). Furthermore the identity of K-functions follows,

$$K_n^{(T)}(\underline{\theta}) = -\frac{\alpha\left(1 - \frac{\beta^2}{8\pi}\right)}{\beta^2}\left(K_n^+(\underline{\theta}) + K_n^-(\underline{\theta})\right)$$

where the normalizations (32) and (34) have been used which means that

$$\frac{N_n^{(T)}}{N_n^+} = 2\frac{\alpha}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right)$$

(3) The claim follows since we have the correspondence of operators and p-functions,

$$\partial_\mu J_L^\mu \leftrightarrow P^+ p_n^{(L,-)}(\underline{\theta}, \underline{l}) + P^- p_n^{(L,+)}(\underline{\theta}, \underline{l}) = -N_n^{(J_L)} m$$

$$\times\left(\sum_{i=1}^n \mathrm{e}^{\theta_i}\sum_{i=1}^n \mathrm{e}^{-\theta_i} - \sum_{i=1}^n \mathrm{e}^{-\theta_i}\sum_{i=1}^n \mathrm{e}^{\theta_i}\right)\sum_{i=1}^n \mathrm{e}^{L(\theta_i - \frac{\mathrm{i}\pi}{2}(1-(-1)^{l_i}\nu))} = 0. \qquad \Box$$

The factor $\frac{\pi\nu}{\sin\pi\nu}$ modifies the classical equation and has to be considered as a quantum correction. For the sinh-Gordon model, an analogous quantum field equation has been obtained in [22][11]. Note that in particular at the 'free fermion point' $\nu \to 1$ ($\beta^2 \to 4\pi$) this factor diverges, a phenomenon which is to be expected from short-distance investigations [36]. For fixed bare mass square α and $\nu \to 2, 3, 4, \dots$ the physical mass goes to zero. These values of the coupling are known to be special: (1) the Bethe ansatz vacuum in the language of

[11] It should be obtained from (37) by the replacement $\beta \to \mathrm{i}g$. However, the relation between the bare mass and the renormalized mass in [22] differs from the analytic continuation of (38).

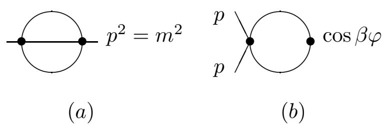


Figure 2. Feynman graphs.

the massive Thirring model shows phase transitions [37] and (2) the model at these points is related [38–40] to Baxters RSOS-models which correspond to minimal conformal models with central charge $c = 1 - 6/(\nu(\nu+1))$ (see also [22]).

The second formula (39) is consistent with renormalization group arguments [41, 42]. In particular, this means that $\beta^2/4\pi$ is the anomalous dimension of $\cos\beta\varphi$. Again this operator equation is modified by a quantum correction $(1 - \beta^2/8\pi)$. Obviously, for fixed bare mass square α and $\beta^2 \to 8\pi$ the model will become conformal invariant. This in turn is related to a Berezinski–Kosterlitz–Thouless phase transition [9, 43, 44]. The conservation law $\partial_\mu T^{\mu\nu} = 0$ for the energy–momentum tensor holds, because it is obtained from the higher currents for $L = \pm 1$. All the results may be checked in perturbation theory by Feynman graph expansions. In particular, in lowest order, the relation between the bare and the renormalized mass (38) is given by figure 2(a). It has already been calculated in [14] and yields

$$m^2 = \alpha\left(1 - \frac{1}{6}\left(\frac{\beta^2}{8}\right)^2 + O(\beta^6)\right)$$

which agrees with the exact formula above. Similarly, we check the quantum corrections of the trace of the energy–momentum tensor (39) by calculating the Feynman graph of figure 2(b) with the result again taken from [14] as

$$\langle p|{:}\cos\beta\varphi - 1{:}|p\rangle = -\beta^2\left(1 + \frac{\beta^2}{8\pi}\right) + O(\beta^6).$$

This again agrees with the exact formula above since the normalization given by equation (31) implies $\langle p|T^\mu_\mu|p\rangle = 2m^2$.

6. Other representations of form factors

6.1. Determinant representation of bosonic sine-Gordon form factors

The scaling Lee–Yang model is equivalent to the breather part of the sine-Gordon model for the coupling constant equal to $\nu = \frac{1}{3}$ (in our notation). For this model, Smirnov [17] derived a determinant formula for form factors (see also [19]). Generalizing this formula in [20, 21] form factors were proposed for the sinh-Gordon model in terms of determinants. The sinh-Gordon model form factors should be identified with sine-Gordon form factors by analytic continuation $\nu \to$ negative values. Using this one would propose for the sine-Gordon model an analogous determinant representation for the K-function of exponentials of the field $\mathcal{O} =: \mathrm{e}^{\mathrm{i}k\beta\varphi}:$,

$$\tilde{K}_n^{(q)}(\underline{\theta}) = (q' - 1/q')^n \prod_{i,j=1}^{n} (x_i + x_j)^{-1}\,\mathrm{Det}_n(\underline{x}, k)$$

where $\underline{x} = (x_1, \ldots, x_n)$ with $x_i = \mathrm{e}^{\theta_i}$ and $q' = \mathrm{e}^{\mathrm{i}\pi\nu}$. The determinant is

$$\mathrm{Det}_n(\underline{x}, k) = \det\left(((k+i-j)_{q'}\sigma_{2i-j-1}(\underline{x}))_{i,j=1}^{n}\right)$$
$$= \begin{vmatrix} (k)_{q'}\sigma_0 & \cdots & (k-n+1)_{q'}\sigma_{-n+1} \\ \vdots & \ddots & \vdots \\ (k+n-1)_{q'}\sigma_{2n-2} & \cdots & (k)_{q'}\sigma_{n-1} \end{vmatrix}$$

where the symmetric polynomials $\sigma_l(\underline{x})$ are defined by

$$\prod_{l=1}^{n}(y + x_l) = \sum_{\lambda=0}^{n} y^{n-\lambda}\sigma_\lambda^{(n)}(\underline{x})$$

and $(k)_{q'} = \sin k\pi\nu / \sin\pi\nu$. This relation of $\tilde{K}_n^{(q)}(\underline{\theta})$ with $\mathrm{Det}_n(\underline{x}, k)$ could be proved similar as in the proof of lemma 4, once the corresponding recursion relation (11) has been proved for $\mathrm{Det}_n(\underline{x}, k)$. This has been done only for the special values $\nu = -\frac{1}{2}, -\frac{1}{3}$ of the coupling constant [20].

6.2. Integral representations of breather form factors

In [1, 2] integral representations for solitonic form factors were proposed. These formulae are quite general and model independent, so analogous formulae should also hold for breather form factors. We propose for n lowest breathers and $0 \leqslant m \leqslant n$

$$\mathcal{O}_n(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} \mathrm{d}z_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} \mathrm{d}z_m h(\underline{\theta}, \underline{z}) \check{p}^{\mathcal{O}}(\underline{\theta}, \underline{z}) \tag{40}$$

with the scalar function (cf [1])

$$h(\underline{\theta}, \underline{z}) = \prod_{1 \leqslant i < j \leqslant n} F(\theta_{ij}) \prod_{i=1}^{n}\prod_{j=1}^{m} \tilde{\phi}(\theta_i - z_j) \prod_{1 \leqslant i < j \leqslant m} \tau(z_i - z_j)$$

and

$$\tilde{\phi}(z) = \frac{S(z)}{F(z)F(z+\mathrm{i}\pi)} = 1 + \frac{\mathrm{i}\sin\pi\nu}{\sinh z} = \frac{\sinh z + \mathrm{i}\sin\pi\nu}{\sinh z}.$$

$$\tau(z) = \frac{1}{\tilde{\phi}(z)\tilde{\phi}(-z)} = \frac{\sinh^2 z}{\sinh^2 z + \sin^2\pi\nu}.$$

The two breather form factor function $F(\theta)$ is again defined by equation (4). For all integration variables z_j $(j = 1, \ldots, m)$ the integration contours $\mathcal{C}_{\underline{\theta}}$ enclose clockwise oriented points $z_j = \theta_i$ $(i = 1, \ldots, n)$. The above integral representation satisfies all form factor properties if suitable conditions for the new type of p-function[12] $\check{p}^{\mathcal{O}}(\theta, z)$ are assumed. Here we consider $\check{p} =$ constant. The form factors of the exponential of the field $\mathcal{O}(x) =: \mathrm{e}^{\mathrm{i}\gamma\varphi} : (x)$ are given by linear combinations of expressions (40),

$$\mathcal{O}_n(\underline{\theta}) = \left(\sqrt{Z^{\varphi}}\frac{\beta}{2\pi\nu}\right)^n \prod_{1 \leqslant i < j \leqslant n} F(\theta_{ij}) \sum_{m=0}^{n} q^{n-2m}(-1)^m I_{nm}(\underline{\theta}) \tag{41}$$

[12] A similar function was used by Cardy and Mussardo [18] in the case of the scaling Ising model to represent the various operators.

where again $q = \exp\left(\mathrm{i}\frac{\pi\nu}{\beta}\gamma\right)$ and

$$I_{nm}(\underline{\theta}) = \frac{1}{m!}\int_{\mathcal{C}_{\underline{\theta}}}\frac{\mathrm{d}z_1}{R}\cdots\int_{\mathcal{C}_{\underline{\theta}}}\frac{\mathrm{d}z_m}{R}\prod_{i=1}^{n}\prod_{j=1}^{m}\tilde{\phi}(\theta_i - z_j)\prod_{1\leqslant i<j\leqslant m}\tau(z_i - z_j)$$
$$= \sum_{\substack{K\subseteq N\\ |K|=m}}\prod_{i\in N\setminus K}\prod_{k\in K}\tilde{\phi}(\theta_{ik})$$

with $R = \operatorname*{Res}_{\theta=0}\tilde{\phi}(\theta)$ and $N = \{1,\ldots,n\}$, $K = \{k_1,\ldots,k_m\}$. It is easy to verify that the asymptotic behaviour (20) is satisfied. Also the low particle number form factors agree with equations (24). This proves formula (41).

Another integral representation is directly obtained from the integral representations for solitonic form factors in [2],

$$\mathcal{O}(\underline{\theta}) = \tilde{N}_n^{\mathcal{O}}\prod_{i<j}\left(F(\theta_{ij})\tanh\frac{1}{2}\theta_{ij}\sinh\frac{1}{2}(\theta_{ij}+\mathrm{i}\pi\nu)\sinh\frac{1}{2}(\theta_{ij}-\mathrm{i}\pi\nu)\right)$$
$$\times\int_{\mathcal{D}_{\theta_1}}\mathrm{d}z_1\cdots\int_{\mathcal{D}_{\theta_n}}\mathrm{d}z_r\prod_{i=1}^{n}\prod_{j=1}^{n}\chi(\theta_i - z_j)\prod_{i<j}\sinh z_{ij}\,p(\underline{\theta},\underline{z}) \tag{42}$$

where the contour $\mathcal{D}_{\theta_i}$ consists of two circles around the poles at $\theta_i - \frac{\mathrm{i}\pi}{2}(1\pm\nu)$ and

$$\chi(\theta) = \frac{1}{\sinh\frac{1}{2}\left(\theta - \frac{\mathrm{i}\pi}{2}(1-\nu)\right)\sinh\frac{1}{2}\left(\theta - \frac{\mathrm{i}\pi}{2}(1+\nu)\right)}.$$

As a matter of fact, in [2] from this integral representation the representation (1) with the K-function (2) was derived using the identity

$$\sinh\tfrac{1}{2}(\theta_{ij}-\mathrm{i}\pi\nu)\chi\left(\theta_i - z_j^{(l_j)}\right)\chi\left(\theta_j - z_i^{(l_i)}\right)\sinh\left(z_i^{(l_i)} - z_j^{(l_j)}\right)$$
$$= \frac{2}{\tanh\frac{1}{2}\xi_{ij}\sinh\frac{1}{2}(\xi_{ij}+\mathrm{i}\pi\nu)\sinh\frac{1}{2}(\xi_{ij}-\mathrm{i}\pi\nu)}\left(1 + (l_i - l_j)\frac{\mathrm{i}\sin\pi\nu}{\sinh\xi_{ij}}\right)$$

for $l_i, l_j = 0, 1$ and $z_i^{(l_i)} = \xi_i - \frac{\mathrm{i}\pi}{2}(1-(-1)^{l_i}\nu)$. Performing one integration in equation (42) and using symmetry properties of the integrand, one obtains an integral representation of the type as used by Smirnov in [17].

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Appendix. Asymptotic behaviour

Lemma 6. *The K-functions defined by equation (2) and the p-functions*

$$\text{(a)}\quad \tilde{p}_n^{(q)}(\underline{\theta}) = \prod_{i=1}^{n} q^{(-1)^{l_i}}$$

$$\text{(b)}\quad \tilde{p}_n^{(N)}(\underline{\theta}) = \left(\sum_{i=1}^{n}(-1)^{l_i}\right)^N$$

$$\text{(c)}\quad \tilde{p}_n^{(\pm)}(\underline{\theta}) = \sum_{i=1}^{n} \mathrm{e}^{\mp\theta_i} \sum_{i=1}^{n} \mathrm{e}^{\pm z_i^{(l_i)}}$$

satisfy for $\operatorname{Re}\theta_1 \to \infty$ *the asymptotic behaviour*

$$\text{(a)}\quad \tilde{K}_n^{(q)}(\underline{\theta}) = \tilde{K}_1^{(q)}(\theta_1)\tilde{K}_{n-1}^{(q)}(\underline{\theta}') + O(\mathrm{e}^{-\operatorname{Re}\theta_1})$$

$$\text{(b)}\quad \tilde{K}_n^{(N)}(\underline{\theta}) = \sum_{K=1}^{N-1} \binom{N}{K} \tilde{K}_1^{(K)}(\theta_1)\tilde{K}_{n-1}^{(N-K)}(\underline{\theta}') + O(\mathrm{e}^{-\operatorname{Re}\theta_1})$$

$$\text{(c)}\quad \tilde{K}_n^{\pm}(\underline{\theta}) = \pm 2\mathrm{i}\sin\pi\nu\, \tilde{K}_{n-1}^{\pm}(\underline{\theta}') + O(\mathrm{e}^{-\operatorname{Re}\theta_1})$$

with $\underline{\theta}' = (\theta_2, \ldots, \theta_1)$. *In particular,*

$$\tilde{K}_1^{(1)}(\theta) = \text{const} \qquad \textit{and} \qquad \tilde{K}_n^{(1)}(\underline{\theta}) = O(\mathrm{e}^{-\operatorname{Re}\theta_1}) \qquad \textit{for} \quad n > 1.$$

Proof. The first two asymptotic relations are quite obvious. Note that $\tilde{K}_1^{(q)} = q - 1/q$ and $\tilde{K}_1^{(1)}(\underline{\theta}) = 2$.

(a) For $\tilde{K}_n^{(q)}(\underline{\theta})$ and $\operatorname{Re}\theta_1 \to \infty$ we have

$$\tilde{K}_n^{(q)}(\underline{\theta}) = \sum_{l_1=0}^{1}(-1)^{l_1} q^{(-1)^{l_1}} \tilde{K}_{n-1}^{(q)}(\underline{\theta}') + O(\mathrm{e}^{-\operatorname{Re}\theta_1})$$

(b) For $\tilde{K}_n^{(N)}(\underline{\theta})$ we use

$$\left(\sum_{i=1}^{n}(-1)^{l_i}\right)^N = \sum_{K=0}^{N}\binom{N}{K}((-1)^{l_1})^K \left(\sum_{i=2}^{n}(-1)^{l_i}\right)^{N-K}$$

which proves the claim as in the previous case.

(c) For $\tilde{K}_n^{+}(\underline{\theta})$ and $\operatorname{Re}\theta_1 \to \infty$ we have

$$\begin{aligned}\sum_{i=1}^{n}\mathrm{e}^{-\theta_i}\sum_{i=1}^{n}\mathrm{e}^{z_i^{(l_i)}} &= \left(\mathrm{e}^{-\theta_1} + \sum_{i=2}^{n}\mathrm{e}^{-\theta_i}\right)\left(\mathrm{e}^{\theta_1 - \frac{\mathrm{i}\pi}{2}(1-(-1)^{l_1}\nu)} + \sum_{i=2}^{n}\mathrm{e}^{z_i^{(l_i)}}\right)\\ &= \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}(1-(-1)^{l_1}\nu)} + \left(\sum_{i=2}^{n}\mathrm{e}^{-\theta_i}\right)\mathrm{e}^{\theta_1 - \frac{\mathrm{i}\pi}{2}(1-(-1)^{l_1}\nu)}\\ &\quad + \sum_{i=2}^{n}\mathrm{e}^{-\theta_i}\sum_{i=2}^{n}\mathrm{e}^{z_i^{(l_i)}} + O(\mathrm{e}^{-\operatorname{Re}\theta_1}).\end{aligned}$$

We calculate the leading terms $O(1)$. The contribution of the first term consists of two types: one vanishes because of the lemma above and the other is of order $O(\mathrm{e}^{-\operatorname{Re}\theta_1})$.

The contribution of the third term vanishes after summation over l_1. The contribution of the second term is proportional to

$$\sum_{l_1=0}^{1}(-1)^{l_1}\left(1+\sum_{j=2}^{n}(l_1-l_j)\frac{\mathrm{i}\sin\pi\nu}{\sinh\theta_{1j}}\right)\mathrm{e}^{\theta_1-\frac{\mathrm{i}\pi}{2}(1-(-1)^{l_1}\nu)}$$

$$\approx -\mathrm{i}\mathrm{e}^{\theta_1}\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\nu}-\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\nu}\right)+2\mathrm{i}\sin\pi\nu\sum_{j=2}^{n}\sum_{l_1=0}^{1}(-1)^{l_1}(l_1-l_j)\,\mathrm{e}^{\theta_j-\frac{\mathrm{i}\pi}{2}(1-(-1)^{l_1}\nu)}.$$

The first term again vanishes due to the lemma and the second one yields

$$\begin{aligned}
&2\mathrm{i}\sin\pi\nu\sum_{j=2}^{n}\sum_{l_1=0}^{1}(-1)^{l_1}(l_1-l_j)\,\mathrm{e}^{\theta_j-\frac{\mathrm{i}\pi}{2}(1-(-1)^{l_1}\nu)}\\
&\qquad=2\mathrm{i}\sin\pi\nu\sum_{j=2}^{n}\sum_{l_1=0}^{1}(-1)^{l_1}(l_1-l_j)\,\mathrm{e}^{\theta_j-\frac{\mathrm{i}\pi}{2}(1-(-1)^{1-l_j}\nu)}\\
&\qquad=-2\mathrm{i}\sin\pi\nu\sum_{j=2}^{n}\mathrm{e}^{\theta_j-\frac{\mathrm{i}\pi}{2}(1+(-1)^{l_j}\nu)}.
\end{aligned}$$

Therefore, we finally obtain the asymptotic behaviour

$$\begin{aligned}
\tilde{K}_n^+(\underline{\theta})&=-2\mathrm{i}\sin\pi\nu\sum_{l_2=0}^{1}\cdots\sum_{l_n=0}^{1}(-1)^{l_2+\cdots+l_n}\prod_{2\leqslant i<j\leqslant n}\left(1+(l_i-l_j)\frac{\mathrm{i}\sin\pi\nu}{\sinh\theta_{ij}}\right)\\
&\qquad\times\sum_{i=2}^{n}\mathrm{e}^{-\theta_i}\sum_{j=2}^{n}\mathrm{e}^{\theta_j-\frac{\mathrm{i}\pi}{2}(1+(-1)^{l_j}\nu)}+O(\mathrm{e}^{-\mathrm{Re}\,\theta_1})\\
&=2\mathrm{i}\sin\pi\nu\,\tilde{K}_{n-1}^+(\underline{\theta'})+O(\mathrm{e}^{-\mathrm{Re}\,\theta_1})
\end{aligned}$$

with $\underline{\theta'}=(\theta_2,\ldots,\theta_n)$. We have used

$$\begin{aligned}
&\sum_{l_2=0}^{1}\cdots\sum_{l_n=0}^{1}(-1)^{l_2+\cdots+l_n}\prod_{2\leqslant i<j\leqslant n}\left(1+(l_i-l_j)\frac{\mathrm{i}\sin\pi\nu}{\sinh\theta_{ij}}\right)\\
&\qquad\times\sum_{j=2}^{n}\left(\mathrm{e}^{\theta_j-\frac{\mathrm{i}\pi}{2}(1+(-1)^{l_j}\nu)}+\mathrm{e}^{\theta_j-\frac{\mathrm{i}\pi}{2}(1-(-1)^{l_j}\nu)}\right)=0
\end{aligned}$$

which follows from lemma 2. For $\tilde{K}_n^-(\underline{\theta})$ and $\mathrm{Re}\,\theta_1\to\infty$ we have

$$\begin{aligned}
\sum_{i=1}^{n}\mathrm{e}^{\theta_i}\sum_{i=1}^{n}\mathrm{e}^{-z_i^{(l_i)}}&=\left(\mathrm{e}^{\theta_1}+\sum_{i=2}^{n}\mathrm{e}^{\theta_i}\right)\left(\mathrm{e}^{-\theta_1+\frac{\mathrm{i}\pi}{2}(1-(-1)^{l_1}\nu)}+\sum_{i=2}^{n}\mathrm{e}^{-z_i^{(l_i)}}\right)\\
&=\mathrm{e}^{\frac{\mathrm{i}\pi}{2}(1-(-1)^{l_1}\nu)}+\mathrm{e}^{\theta_1}\sum_{i=2}^{n}\mathrm{e}^{-\theta_j-\frac{\mathrm{i}\pi}{2}(1-(-1)^{l_j}\nu)}+\sum_{i=2}^{n}\mathrm{e}^{\theta_i}\sum_{i=2}^{n}\mathrm{e}^{-z_i^{(l_i)}}+O(\mathrm{e}^{-\mathrm{Re}\,\theta_1}).
\end{aligned}$$

We again calculate the leading terms $O(1)$. Again the contribution of the first term consists of two types: one vanishes because of the lemma above and the other is of order $O(\mathrm{e}^{-\mathrm{Re}\,\theta_1})$.

The contribution of the third term vanishes after summation over l_1. The contribution of the second term is proportional to

$$\sum_{l_1=0}^{1}(-1)^{l_1}\left(1+\sum_{j=2}^{n}(l_1-l_i)\frac{\mathrm{i}\sin\pi\nu}{\sinh\theta_{1i}}\right)\mathrm{e}^{\theta_1}\sum_{j=2}^{n}\mathrm{e}^{-\theta_j+\frac{\mathrm{i}\pi}{2}(1-(-1)^{l_j}\nu)}$$
$$\approx 2\mathrm{i}\sin\pi\nu\sum_{j=2}^{n}\mathrm{e}^{\theta_i}\sum_{l_1=0}^{1}(-1)^{l_1}(l_1-l_j)\,\mathrm{e}^{-\theta_j+\frac{\mathrm{i}\pi}{2}(1-(-1)^{l_j}\nu)}$$
$$=-2\mathrm{i}\sin\pi\nu\sum_{j=2}^{n}\mathrm{e}^{\theta_i}\sum_{j=2}^{n}\mathrm{e}^{-\theta_j+\frac{\mathrm{i}\pi}{2}(1-(-1)^{l_j}\nu)}.$$

Therefore, we finally obtain the asymptotic behaviour

$$\tilde{K}_n^-(\underline{\theta})=-2\mathrm{i}\sin\pi\nu\sum_{l_2=0}^{1}\dots\sum_{l_n=0}^{1}(-1)^{l_2+\cdots+l_n}\prod_{2\leqslant i<j\leqslant n}\left(1+(l_i-l_j)\frac{\mathrm{i}\sin\pi\nu}{\sinh\theta_{ij}}\right)$$
$$\times\sum_{i=2}^{n}\mathrm{e}^{\theta_i}\sum_{j=2}^{n}\mathrm{e}^{-\theta_j+\frac{\mathrm{i}\pi}{2}(1+(-1)^{l_j}\nu)}+O(\mathrm{e}^{-\mathrm{Re}\,\theta_1})$$
$$=-2\mathrm{i}\sin\pi\nu\,\tilde{K}_{n-1}^-(\underline{\theta}')+O(\mathrm{e}^{-\mathrm{Re}\,\theta_1}).\qquad\square$$

Analogously, one may discuss the behaviour for $\mathrm{Re}\,\theta_1\to-\infty$.

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